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Velocity and concentration distributions of round and plane turbulent jets

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Abstract. The analytical solutions for the velocity distribution of turbulent round and plane jets described in the classical textbook of Schlichting (*Boundary Layer Theory* 7th edition (1979)) based on boundary-layer approximations are well-known, and have provided the fundamental understanding of the mechanics of turbulent jets. However, the scaling coefficients involved were not well quantified as discussed recently by Mathieu and Scott (*An Introduction to Turbulent Flow* (2000)). In this paper, it is shown that the coefficients can be better determined by the available experimental measurements in the literature. Furthermore, by assuming that the turbulent diffusivity relates to the eddy viscosity, it is shown that closed-form analytical solutions can also be obtained for the scalar concentration distribution in addition to the velocity distribution. The turbulent Schmidt number is found to be less than 1 for both plane and round jets, and close to the isotropic turbulence value of 0·7 in the round jet case.

Key words: analytical solutions, eddy viscosity, turbulent diffusivity, turbulent jets

1. Introduction

Turbulent round and plane jets in stagnant ambient have been studied extensively as basic representations of turbulent shear flow in fluid mechanics. They are also rudimentary scenarios for the dilution analysis of wastewater effluent discharges. Experimental investigations in the past have revealed in detail their characteristics such as the decay and cross-sectional distribution of the axial velocity and scalar concentration. A good summary can be found in [1, Chapter 9].

Turbulent jets are essentially boundary-layer phenomena due to the slender aspect ratios. With the boundary-layer approximations and the assumptions of self-similarity, analytical solutions for the velocity distribution profile for both round and plane jets had been obtained with the eddy-viscosity approach. These solutions, described in the classical text of [2, Chapter 14], are well-known and have provided the fundamental understanding of the mechanics of turbulent jets. They have been commonly presented in advanced textbooks in fluid mechanics, *e.g.* [3, Chapter 5], [4, Chapter 5], [5, Chapter 6]. The self-similarity assumptions have also been well verified in experimental studies conducted with advanced measurement techniques as presented in [6–8].

Despite the general good fit of the analytical solutions to the experimental data in terms of the non-dimensional profiles, a major problem exists due to the fact that the scaling coefficients involved are not well quantified as addressed in [3] and also discussed recently in [4]. In this study, we attempt to define these scaling coefficients deterministically based on conservation laws together with the available experimental measurements in the literature. Furthermore, by assuming that the turbulent diffusivity bears a ratio to the eddy viscosity, which is a common assumption adopted in many commercial CFD codes closed-form analytical solutions are derived for the scalar concentration distribution in addition to the velocity distribution. Although similarity solutions for the concentration distribution for turbulent jets have been explored in the literature (*e.g.* [9]), the closed-form analytical solutions derived here have not been previously reported as far as the author is aware.

2. Round jets

We begin with the derivation for a turbulent round jet following the procedures of [2], as well as the recent additions reported in [3], and [4]. The conservation of mass in the axisymetric coordinate system is given by:

$$\frac{1}{r}\frac{\partial r\,\bar{v}}{\partial r}+\frac{\partial \bar{w}}{\partial z}=0, \tag{1}$$

where $\bar{v}$ and $\bar{w}$ are the mean velocity in the radial r- and axial z-directions respectively. The Reynolds-averaged Navier–Stokes equation with the boundary-layer approximations for a three-dimensional round jet can be expressed as [2]:

$$\bar{w}\frac{\partial \bar{w}}{\partial z}+\bar{v}\frac{\partial \bar{w}}{\partial r}=-\frac{1}{r}\frac{\partial r\overline{w'v'}}{\partial r}, \tag{2}$$

where v' and w' are the velocity fluctuations in the r- and z-directions, respectively. Integration of (2) with respect to r shows that the mean momentum, M_M, is conserved with distance z, *i.e.*:

$$\frac{\partial\, M_M}{\partial\, z}=\frac{\partial}{\partial z}\int_0^\infty 2\pi\,\bar{w}^2 r\,\mathrm{d}r=0. \tag{3}$$

We now derive the velocity-distribution functions in a manner that follows closely the previous references, particularly [4]. However, instead of working with dimensional distribution functions, as in the references, the distribution functions introduced here have been made non-dimensional observing the known facts that they should be proportional with the square root of the initial momentum, M_o, and inversely proportional with the distance, z, *i.e.*

$$\bar{w}=\frac{M_o^{1/2} f\,(\xi)}{(z-z_o)},\qquad \bar{v}=\frac{M_o^{1/2} g\,(\xi)}{(z-z_o)}, \tag{4, 5}$$

where z_o is the virtual origin, and $\xi=\frac{r}{(z-z_o)}$ a non-dimensional radial variable. For a round jet, the initial momentum is given by:

$$M_o=\frac{\pi}{4}D^2 w_o^2, \tag{6}$$

where D is the nozzle diameter and w_o is the initial velocity.

Substituting (4) and (5) in (1) and integrating, we have:

$$\xi g\,(\xi)=\xi^2\, f\,(\xi)-F\,(\xi)\,;\quad F\,(\xi)=\int_0^\xi \xi' f\left(\xi'\right)\mathrm{d}\xi'. \tag{7}$$

A closure is needed for the Reynolds-stress term in the RHS of (2). The basic eddy-viscosity representation is adopted, namely that:

$$\overline{w'v'}=-\nu_t\frac{\partial \bar{w}}{\partial r}, \tag{8}$$

where ν_t is the eddy viscosity that does not depend on the transverse direction (see [4, Chapter 5] for a detailed discussion), i.e.

$$\nu_t = \lambda M_o^{1/2}. \tag{9}$$

Substituting (6) and (8) in (2) and solving as in [3] and [4], we have

$$F = \frac{4\lambda\,(\alpha\xi)^2}{1+(\alpha\xi)^2}. \tag{10}$$

where α is an integration constant that represents a scale of the width. With F known, the velocity-distribution functions can be resolved as:

$$f = \frac{8\lambda\alpha^2}{\left(1+(\alpha\xi)^2\right)^2}, \qquad g = \frac{4\lambda\alpha^2\xi\left(1+(\alpha\xi)^2\right)}{\left(1+(\alpha\xi)^2\right)^2}. \tag{11, 12}$$

The meaning of α as a scaling constant for the jet width is well understood, but its quantitative value has not been defined in the references. It is commented, in [4, Chapter 5], that α is not a fixed value but rather can be looked upon as a fitting constant, and a value of 8 is suggested based on the matching of the observed experimental velocity profile. Here, we attempt to quantify α in a more physical manner by employing the conservation of the mean momentum flux. Given (11), the mean momentum flux can be calculated as:

$$M_M = \left(\int_0^\infty 2\pi\xi f^2(\xi)\,\mathrm{d}\xi\right) M_o = \frac{64}{3}\pi\alpha^2\lambda^2 M_o. \tag{13}$$

Recently, the relationship between the total and mean momentum fluxes for a round jet was investigated experimentally by [8]. It was found that the two fluxes are related by a constant ratio k_{jM} as

$$M_o = k_{jM} M_M \tag{14}$$

and that k_{jM} is equal to 1·10 independent of z. Note that, since the total momentum flux should be conserved for a free jet, it is replaced by the source momentum flux M_o in (14). Equations (13) and (14) together imply that

$$\lambda = \frac{\sqrt{3}}{8\pi^{\frac{1}{2}}\alpha k_{jM}^{\frac{1}{2}}}. \tag{15}$$

At the centerline,

$$f(\xi = 0) = 8\alpha^2\lambda = \sqrt{\frac{4}{\pi}}\,k_{jw} \tag{16}$$

where k_{jw} is the centerline-decay coefficient commonly expressed in the following format in the literature:

$$\frac{w_o}{\bar{w}_c} = \frac{(z - z_o)}{k_{jw} D}, \tag{17}$$

where $\bar{w}_c$ is the centerline velocity. The coefficient k_{jw} has been investigated extensively in the past ([1]). The recent investigation in [8] shows a value of 6·48 by taking $z_o = 0$ which should be reasonable, given the fact that the virtual origin of a turbulent round jet is less than one nozzle diameter. Solving the two simultaneous equations (15) and (16), we have

$$\alpha = \frac{2k_{jM}^{\frac{1}{2}} k_{jw}}{\sqrt{3}} \qquad \text{and} \qquad \lambda = \frac{3}{16\sqrt{\pi} k_{jM} k_{jw}}. \tag{18a,b}$$

Substitution of the above-suggested values for k_{jM} and k_{jw} yields $\alpha = 7{\cdot}84$ and $\lambda = 0{\cdot}0149$. Figure 1 shows the predicted non-dimensional velocity distribution (scaled with the centerline velocity), in comparison with the experimental measurements taken from [8]. Clearly, the matching is quite satisfactory, suggesting that the basic eddy-viscosity approach gives a good representation of the underlying mechanics.

We will now continue and analyze the distribution of the scalar concentration in a similar manner. With the boundary-layer approximations, the conservation equation for the scalar concentration can be expressed as:

$$\bar{w}\frac{\partial \bar{c}}{\partial z} + \bar{v}\frac{\partial \bar{c}}{\partial r} = -\frac{1}{r}\frac{\partial r\,\overline{c'v'}}{\partial r}, \tag{19}$$

where $\bar{c}$ is the mean concentration and c' is the concentration fluctuation. We introduce a non-dimensional scalar distribution function as follows:

$$\bar{c} = \frac{Q_o C_o \chi(\xi)}{M_o^{1/2}(z - z_o)}. \tag{20}$$

Adopting a turbulent diffusivity D_t that is similar to the eddy diffusivity, we can model the turbulence mass flux in (19) as:

$$\overline{c'v'} = -D_t \frac{\partial \bar{c}}{\partial r}. \tag{21}$$

Furthermore, D_t and ν_t are assumed to be related by a constant σ,

$$D_t = \sigma \nu_t, \qquad \sigma = \frac{1}{\mathrm{Sc}_t}, \tag{22a,b}$$

where Sc_t is the turbulent Schmidt number. Equation (22) is an assumption commonly adopted nowadays in many commercial CFD codes for the turbulence closure of scalar transport, and is shown to be valid in isotropic turbulence ([10, Chapter 4]). Substituting (21) and (22) in (19) and simplifying, we have

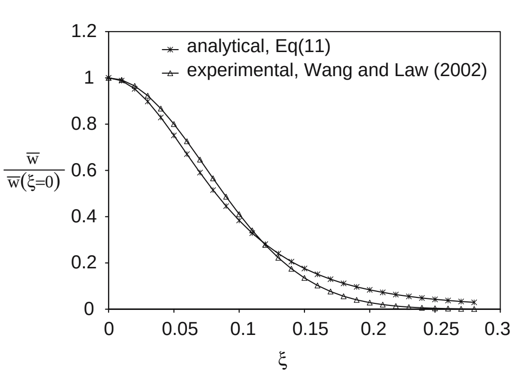

Figure 1. Velocity distribution for a round jet.

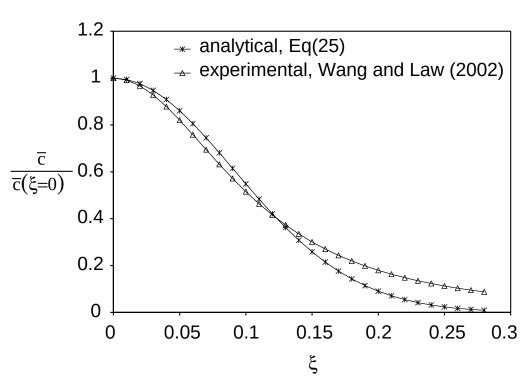

Figure 2. Concentration distribution for a round jet.

$$\sigma\lambda\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\chi}{\mathrm{d}\xi}\right)+F\frac{\mathrm{d}\chi}{\mathrm{d}\xi}+\xi f\chi=0. \tag{23}$$

With the velocity-distribution functions defined in (10) and (11), we have

$$\sigma\xi\left(1+\alpha^2\zeta^2\right)^2\frac{\mathrm{d}^2\chi}{\mathrm{d}\xi^2}+\left(\sigma\left(1+\alpha^2\zeta^2\right)^2+4\alpha^2\zeta^2\left(1+\alpha^2\zeta^2\right)\right)\frac{\mathrm{d}\chi}{\mathrm{d}\xi}+8\alpha^2\xi\chi=0. \tag{24}$$

The general solution to (24) that enforces a monotonic decrease from the centerline is:

$$\chi=\frac{K_R}{\left(1+(\alpha\xi)^2\right)^{2/\sigma}}, \tag{25}$$

where K_R is a scalar decay constant. At the centerline, the scalar concentration from (20) and (25) is:

$$\bar{c}_c=\frac{K_R Q_o C_o}{M_o^{1/2}\,(z-z_o)}, \tag{26}$$

where Q_o is the discharge and C_o the source concentration. Experimentally, it has been found that the centerline decay of scalar concentration follows the format of:

$$\bar{c}_c=\frac{k_{jc}C_o D}{(z-z_o)}, \tag{27}$$

where k_{jc} is the decay constant ([1]). The experimental investigation in [8] showed that $k_{jc}=5{\cdot}26$ by taking the virtual origin to be zero. Solving (26) and (27), we obtain

$$K_R=\frac{2k_{jc}}{\sqrt{\pi}}, \tag{28}$$

i.e.

$$\chi=\frac{2k_{jc}}{\sqrt{\pi}\left(1+(\alpha\xi)^2\right)^{2/\sigma}}. \tag{29}$$

It is also found that ([8]), for a round jet, the mean mass flux is related to the total mass flux by a constant ratio:

$$Q_oC_o=k_{jH}H_{jM}=k_{jH}\left(2\pi\int_0^\infty\bar{w}\,\bar{c}\,r\,\mathrm{d}r\right) \tag{30}$$

where k_{jH} is determined to be 1·076. Solving (30), we obtain

$$\sigma=\frac{2}{16\lambda\pi^{\frac{1}{2}}k_{jc}k_{jH}-1} \tag{31}$$

Substituting the above values yields $\sigma=1{\cdot}45$. In other words, the turbulent diffusivity is larger than the eddy viscosity by 45%, which explains the common observation that the spread for the concentration is wider than the velocity spread for a round jet. The turbulent Schmidt number is equal to $1/1{\cdot}45=0{\cdot}690$ which is very close to the value of 0·7 found in isotropic turbulence ([10, Chapter 4]). Figure 2 shows the non-dimensional scalar concentration distribution compared with the experimental measurements of [8]. The matching is satisfactory up to approximately 0·12, but after that the analytical solution decreases at a faster rate. This shows that the turbulent-diffusivity approach is valid near the jet's centerline, but somewhat overestimates the scalar dispersion away from the center.

3. Plane jets

We shall now derive the analytical solutions for two-dimensional plane jets with the same idealized assumptions. The conservation of mass in the Cartesian coordinate is:

$$\frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{v}}{\partial y} = 0, \tag{32}$$

where z and y are the longitudinal and transverse coordinate, respectively. The Reynolds-averaged Navier–Stokes equation for the two-dimensional plane jet subject to the boundary-layer approximations is:

$$\bar{w}\frac{\partial \bar{w}}{\partial z} + \bar{v}\frac{\partial \bar{w}}{\partial y} = -\frac{\partial \overline{v'w'}}{\partial y}. \tag{33}$$

Integration (33) with respect to y shows that the mean momentum flux M_M is also a constant within this formulation:

$$\frac{\partial M_M}{\partial z} = \frac{\partial}{\partial z}\int_{-\infty}^{\infty} \bar{w}^2 \mathrm{d}y = 0. \tag{34}$$

We now introduce the non-dimensional velocity-distribution functions based on the known observation that the velocity in a plane jet is proportional to the square root of the source momentum flux, M_o, but inversely proportionally to the square root of the distance:

$$\bar{w} = \frac{M_o^{1/2} f(\xi)}{(z - z_o)^{1/2}}, \qquad \bar{v} = \frac{M_o^{1/2} g(\xi)}{(z - z_o)^{1/2}}, \tag{35, 36}$$

where $\xi = \frac{y}{(z - z_o)}$ is a non-dimensional transverse variable for the two-dimensional case. The source momentum flux for a plane jet is defined as:

$$M_o = b\, w_o, \tag{37}$$

where b is the initial slot width. Substitution (35) and (36) in (32) yields:

$$g(\xi) = \xi\, f(\xi) - \frac{1}{2}\int_0^{\xi} f(\xi')\, \mathrm{d}\xi'. \tag{38}$$

The eddy-viscosity assumption is also adopted as:

$$\overline{v'w'} = -\nu_t \frac{\partial \bar{w}}{\partial y} \tag{39}$$

It is further assumed that ν_t depends on the longitudinal but not the transverse direction (see again [4, Chapter 5]) for a detailed description), *i.e.*,

$$\nu_t = \lambda\, M_o^{1/2} (z - z_o)^{1/2}. \tag{40}$$

With Equation (40) we have

$$\lambda \frac{\mathrm{d} f}{\mathrm{d}\xi} + f(\xi) \int_0^{\xi} f(\xi')\, \mathrm{d}\xi' = 0. \tag{41}$$

Integrating (41) and solving as shown in [4], we have

$$F(\xi) = 4\alpha\lambda \tanh(\alpha\xi), \tag{42}$$

where

$$F(\xi) = \int_0^{\xi} f(\xi')\,\mathrm{d}\xi' \tag{43}$$

and α is an integration constant that represents a scale of the jet width. The solution of (42) is:

$$f(\xi) = \frac{4\alpha^2\lambda}{\cosh^2(\alpha\xi)}, \tag{44}$$

$$g(\xi) = \frac{4\alpha^2\lambda\xi}{\cosh^2(\alpha\xi)} - 2\alpha\lambda\tanh(\alpha\xi). \tag{45}$$

We now attempt to quantify the coefficients more precisely using the conservation laws. The mean jet longitudinal momentum flux is calculated as

$$M_M = \left(\int_{-\infty}^{\infty} f^2(\xi)\,\mathrm{d}\xi\right) M_o = \frac{64}{3}\alpha^3\lambda^2 M_o. \tag{46}$$

Assuming that the mean and total momentum fluxes bear a constant ratio as in (14), we observe that α and λ are related as follows:

$$\lambda = \frac{\sqrt{3}}{8\alpha^{\frac{3}{2}} k_{jM}^{\frac{1}{2}}}. \tag{47}$$

At the centerline,

$$f(\xi = 0) = 4\alpha^2\lambda = k_{jw}, \tag{48}$$

where k_{jw} is the centerline decay coefficient in the form of:

$$\frac{w_o}{\bar{w}_c} = \frac{(z - z_o)^{\frac{1}{2}}}{k_{jw} b^{\frac{1}{2}}}, \tag{49}$$

where b is the initial slot width. The parameter k_{jw} was determined, in [1], to be 2·41 by taking the virtual origin to be zero. In other words,

$$\lambda = \frac{k_{jw}}{4\alpha^2}. \tag{50}$$

Solving the two simultaneous equations of (47) and (50) yields:

$$\alpha = \frac{4k_{jM}^2 k_{jw}^2}{3}, \qquad \lambda = \frac{9}{64 k_{jM}^4 k_{jw}^3}. \tag{51a,b}$$

Unlike the round-jet case, the relationship between the total and mean momentum fluxes is not well quantified in the literature for a plane jet. Assuming the same coefficients as in a round jet yields $\alpha = 9{\cdot}37$ and $\lambda = 0{\cdot}00686$. Figure 3 shows a comparison of the non-dimensional velocity distributions. The matching is satisfactory which suggests that the values are reasonable.

The concentration can be analyzed in a similar manner. The conservation equation for the scalar concentration with the boundary-layer approximations can be expressed as:

$$\bar{w}\frac{\partial \bar{c}}{\partial z}+\bar{v}\frac{\partial \bar{c}}{\partial y}=-\frac{\partial \overline{c'v'}}{\partial y}. \tag{52}$$

Define the distribution function for the scalar concentration as

$$\bar{c}=\frac{q_o C_o \chi(\xi)}{M_o^{1/2}(z-z_o)^{1/2}}, \tag{53}$$

where q_o is the source discharge per unit length. Also assume the turbulent diffusivity as

$$\overline{c'v'}=-D_t\frac{\partial \bar{c}}{\partial y} \tag{54}$$

and that D_t and ν_t bear a constant ratio σ as in (22), then from (52)

$$2\sigma\lambda\frac{d^2\chi}{d\xi^2}+F\frac{d\chi}{d\xi}+f\chi=0. \tag{55}$$

Given the known forms of the velocity-distribution functions in (43) and (44), Equation (55) is simplified to:

$$\sigma\cosh^2(\alpha\xi)\frac{d^2\chi}{d\xi^2}+2\alpha\sinh(\alpha\xi)\cosh(\alpha\xi)\frac{d\chi}{d\xi}+2\alpha^2\chi=0. \tag{56}$$

The general solution to (56) that enforces a monotonic decrease from the centerline is:

$$\chi=K_P\,\mathrm{sech}^{\frac{2-\sigma}{2\sigma}}(\alpha\xi)\,\Re\mathfrak{e}\left(\mathrm{P}\left(\frac{2-\sigma}{2\sigma};\frac{2+\sigma}{2\sigma};\mathrm{i}\,\sinh(\alpha\xi)\right)\right), \tag{57}$$

where P( ) is the associated Legendre function of the first kind, $\Re\mathfrak{e}$( ) the real part, $\mathrm{i}=\sqrt{-1}$ and K_P a constant.

At the centre, the function in (57) reduces to

$$\chi(\xi=0)=\chi_c=K_P\ \Re\mathfrak{e}\left(\frac{(2i)^{\frac{2+\sigma}{2\sigma}}}{\Gamma\left(\frac{2-\sigma}{2\sigma}\right)}\right)=K_P\ \frac{(2)^{\frac{2+\sigma}{2\sigma}}}{\Gamma\left(\frac{\sigma-2}{2\sigma}\right)}\cos\left(\frac{(2+\sigma)}{4\sigma}\pi\right), \tag{58}$$

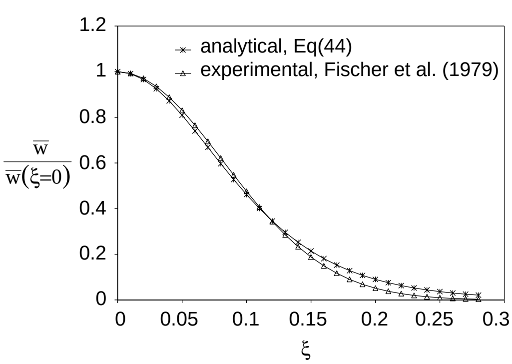

Figure 3. Velocity distribution for a plane jet.

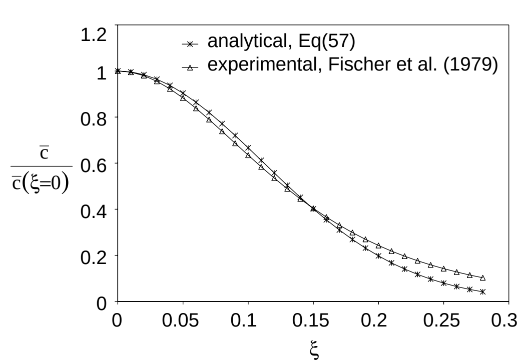

Figure 4. Concentration distribution for a plane jet.

where Γ is the Gamma function. Since $\chi_c = 2{\cdot}38$ from [1], this implies

$$K_P = \frac{2{\cdot}38\ \Gamma\left(\frac{2-\sigma}{2\sigma}\right)}{(2)^{\frac{2+\sigma}{2\sigma}}} \sec\left(\frac{(2+\sigma)}{4\sigma}\pi\right). \tag{59}$$

The mean mass flux can now be computed as

$$q_o C_o = k_{jH} H_{jM} = k_{jH}\left[\int_{-\infty}^{\infty} \bar{w}\,\bar{c}\ \mathrm{d}y\right] \tag{60}$$

or

$$\frac{9{\cdot}56\ (\alpha^2\lambda)\ \Gamma\left(\frac{2-\sigma}{2\sigma}\right)}{(2)^{\frac{2+\sigma}{2\sigma}}} \sec\left(\frac{(2+\sigma)}{4\sigma}\pi\right) \int_{-\infty}^{\infty} \mathrm{sech}^{\frac{2+3\sigma}{2\sigma}}(\alpha\xi)\,\mathfrak{Re}\left(\mathrm{P}\left(\frac{2-\sigma}{2\sigma};\frac{2+\sigma}{2\sigma};\mathrm{i}\ \sinh(\alpha\xi)\right)\right)\mathrm{d}\xi = k_{jH} \tag{61}$$

The integral in (61) does not have a closed-form solution and needs to be resolved numerically. Taking k_{jH} to be 1·073 as in the round jet translates into a value of σ equal to 1·70, or a turbulent Schmidt number of 0·588. Figure 4 shows the comparison on the non-dimensional distribution for the scalar concentration between the predicted and experimental profiles. The comparison is clearly satisfactory, and in fact the matching is somewhat better than in the round jet case.

4. Conclusions

As discussed previously, despite the fact that the analytical solutions of the velocity-distribution profile for turbulent round and plane jets have been known for a long time, a major problem has existed to the present time due to the fact that the scaling coefficients involved are not well quantified. The difficulties in quantifying the values of these scaling coefficients have been discussed recently in [4].

In this paper, we have attempted to determine these scaling coefficients in the analytical solutions based on conservation laws of mass and momentum with the recently available experimental measurements in the literature. Furthermore, by assuming that the turbulent diffusivity bears a constant ratio to the eddy viscosity (which is a common assumption adopted in many modern commercial CFD codes), we have derived closed-form analytical solutions for the scalar concentration distribution in addition to the velocity distribution for both round and plane turbulent jets. As far as the author is aware, these analytical solutions for the scalar concentration have not been presented previously. Taken together, the analytical solutions for both the velocity and concentration distributions provide the basic understanding of the flow and dispersion characteristics by round and plane turbulent jets.

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